Soluții – clasa a XI-a

3. (1) Dezvoltand determinantul dat dupa elementele primei coloanei se obtine

(*) $D_n(x)=(x+2a)D_{n-1}(x)-a(x+a)D_{n-2}(x)$

(2) In (*) avem o …. liniara de ordin 2 in ecuatie caracteristica

$t^2-(x+2a)t+a(x+a)=0$ de radacini

$t_{1,2}=\dfrac{x+2a\pm x}{2}$, deci $t_1=x+a$, $t_2=a$

Determinam $\alpha,\beta \in \Re$ a.i

$D_n(x)=\alpha (x+a)^n+\beta a^n$

Pentru n=1 si n=2 se obtine sistemul liniar in α,β

(**) $\begin{cases} \alpha(x+a)+\beta a = x+2a \\ \alpha(x+a)^2+\beta a^2 = x^2+3ax+3a^2 \end{cases}$ de det $\Delta=-ax(x+a)$. Cand $\Delta\neq 0$, … $x\neq 0$ si $x\neq -a$ se obtine $\alpha=\dfrac{x+a}{x}$, $\beta=-\dfrac{a}{x}$. Asadar din $x\neq 0$ si $x\neq -a$ avem

$D_n(x)=\dfrac{(x+a)^{n+1}-a^{n+1}}{x}=(x+a)^n+(x+a)^{n-1}a+\ldots+(x+a)a^{n-1}+a^n$

4. (1) Fie p, q $\in N^*$ a.i. $A^p=0=B^q$ si fie m=p+q-1. Cum AB=BA, avem

$(A+B)^m=\sum_{k=0}^{m} C_m^k A^{m-k}B^k=0$

pentru ca $m-k\geq p$ sau $k\geq q$ oricare ar fi k=0, 1, 2, …, m

Avem

$e^Ae^B=\left(\sum_{i=0}^{p-1}\frac{1}{i!}A^i\right)\left(\sum_{j=0}^{q-1}\frac{1}{j!}B^j\right)=\sum_{k=0}^{p+q-2}\left(\sum_{i+j=k}\frac{1}{i!\,j!}A^jB^j\right)=\sum_{k=0}^{p+q-2}\frac{1}{k!}\left(\sum_{j=0}^{k}C_k^jA^{k-j}B^j\right)=$

$\sum_{k=0}^{p+q-2}\frac{1}{k!}(A+B)^K=e^{A+B}$

(2) Daca $A^p=0 \Rightarrow (-A)^p=0$ si cum $A(-A)=(-A)A$ rezulta ca $e^Ae^{-A}=e^{A+(-A)}=e^0=I_n$